The higher rank numerical range of matrix polynomials

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Abstract

The notion of the higher rank numerical range $\Lambda_k(L(\lambda))$ for matrix polynomials $L(\lambda) = A_m\lambda^m + \ldots + A_1\lambda + A_0$ is introduced here and some fundamental geometrical properties are investigated. Further, the sharp points of $\Lambda_k(L(\lambda))$ are defined and their relation to the numerical range $w(L(\lambda))$ is presented. A connection of $\Lambda_k(L(\lambda))$ with the vector-valued higher rank numerical range $\Lambda_k(A_0, \ldots, A_m)$ is also discussed.

Key words: higher rank numerical range, matrix polynomials, quantum error correction.
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1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ be the algebra of matrices $A = [a_{ij}]_{i,j=1}^n$ with entries $a_{ij} \in \mathbb{C}$ and

$$L(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$$

be a matrix polynomial with $A_i \in \mathcal{M}_n(\mathbb{C})$ and $A_m \neq 0$. For a positive integer $k \geq 1$, we define the *higher rank numerical range* of $L(\lambda)$ as

$$\Lambda_k(L(\lambda)) = \{\lambda \in \mathbb{C} : PL(\lambda)P = 0_n \ for\ some\ P \in \mathcal{P}_k\}, \tag{1.1}$$

where $\mathcal{P}_k$ is the set of all orthogonal projections P of $\mathbb{C}^n$ onto any k-dimensional subspace $\mathcal{K}$ of $\mathbb{C}^n$. Equivalently,

$$\Lambda_k(L(\lambda)) = \{\lambda \in \mathbb{C} : Q^*L(\lambda)Q = 0_k \ for\ some\ Q \in \mathcal{M}_{n,k}\ with\ Q^*Q = I_k\}, \tag{1.2}$$

since $P = QQ^*$, with $Q \in \mathcal{M}_{n,k}(\mathbb{C})$ and $Q^*Q = I_k$. In case $k = 1$, the set reduces to the well known *numerical range* $w(L(\lambda))$ of $L(\lambda)$ [14]

$$\Lambda_1(L(\lambda)) \equiv w(L(\lambda)) = \{\lambda \in \mathbb{C} : x^*L(\lambda)x = 0 \ for\ some\ x \in \mathbb{C}^n, \|x\| = 1\}. \tag{1.3}$$

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The (1.1) (or (1.2)) is an interesting generalization of numerical range, since matrix polynomials play a significant role in several problems of computational chemistry and structural molecular biology [9]. They consist algebraic tools to computing all conformations of ring molecules and they model various problems in terms of polynomial equations.

If $L(\lambda) = I\lambda - A$, then

$$\begin{aligned}\Lambda_k(I\lambda - A) &= \{\lambda \in \mathbb{C} : PAP = \lambda P \ for\ some\ P \in \mathcal{P}_k\} \\ &= \{\lambda \in \mathbb{C} : Q^*AQ = \lambda I_k\,,\ Q^*Q = I_k,\ Q \in \mathcal{M}_{n,k}(\mathbb{C})\}\,,\end{aligned} \tag{1.4}$$

namely, it coincides with the *higher rank numerical range* of a matrix $A \in \mathcal{M}_n$. The concept of higher rank numerical range has been studied extensively by Choi *et al* in [4, 5, 6, 7] and later by other researchers in [18, 20, 23]. We should note that for $k = 1$, $\Lambda_k(I\lambda - A)$ yields the classical *numerical range* of a matrix A, i.e.

$$F(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\}\,, \tag{1.5}$$

whose basic properties can be found in [10, 11, 12].

A multi-dimensional higher rank numerical range is the *joint higher rank numerical range* [19]

$$\begin{aligned}\Lambda_k(\mathbf{A}) = \big\{(\mu_0, \mu_1, \ldots, \mu_m) \in \mathbb{C}^{m+1} : \exists\ P \in \mathcal{P}_k\ such\ that \\ PA_iP = \mu_i P,\ i = 0, \ldots, m\}\,,\end{aligned} \tag{1.6}$$

where $\mathbf{A} = (A_0, A_1, \ldots, A_m)$ is an $(m + 1)$-tuple of matrices $A_i \in \mathcal{M}_n(\mathbb{C})$ for $i = 0, \ldots, m$. Apparently, for $k = 1$, $\Lambda_1(\mathbf{A})$ is identified with the *joint numerical range*, denoted by

$$w(\mathbf{A}) = \{(x^*A_0x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, \|x\| = 1\}\,. \tag{1.7}$$

In the context of quantum information theory, $\Lambda_k(\mathbf{A})$ is closely related to a quantum error correcting code, since the latter exists as long as the joint higher rank numerical range associated with the error operators of a noisy quantum channel is a non empty set.

In section 2, we present some familiar properties of $\Lambda_k(L(\lambda))$ and we provide a description of the set through intersections of numerical ranges of all compressions of the matrix polynomial $L(\lambda)$ to $(n - k + 1)$-dimensional subspaces. This study originates from an analogous expression for matrices, presented and proved in [1]. It also motivates us to investigate the geometry of $\Lambda_k(L(\lambda))$ proving conditions for its boundedness and elaborating a basic property on the number of its connected components.

In section 3, a connection of the boundary points of $\Lambda_k(L(\lambda))$ with respect to the boundary points of $w(L(\lambda))$ is considered. Particularly, introducing the notion of *sharp points* for $\Lambda_k(L(\lambda))$, we show that a sharp point

of $w(A\lambda - B)$ with algebraic multiplicity k with respect to the spectrum $\sigma(A\lambda - B)$ is also a sharp point of $\Lambda_j(A\lambda - B)$, for $j = 2, \ldots, k$. In section 4, a relationship between $\Lambda_k(L(\lambda))$ and $\Lambda_k(C_L(\lambda))$ is presented, where $C_L(\lambda)$ is the companion polynomial of $L(\lambda)$. Also, we treat a sufficient condition for boundary points of $w(\mathbf{A})$ to be boundary points of $\Lambda_k(\mathbf{A})$, where $\mathbf{A} = (A_0, \ldots, A_m)$ and evenly, we investigate an interplay of $\Lambda_k(L(\lambda))$ and $\Lambda_k(\mathbf{A})$.

2 Geometrical Properties

In the beginning of this section, we present some basic properties as in [20] for the higher rank numerical range of a matrix polynomial $L(\lambda)$.

Proposition 1. *Let $L(\lambda) = \sum_{j=0}^{m} A_j \lambda^j$ be a matrix polynomial, where $A_m \neq 0$, then*

(a) *$\Lambda_k(L(\lambda))$ is closed in $\mathbb{C}$.*

(b) *For any $\alpha \in \mathbb{C}$, $\Lambda_k(L(\lambda + \alpha)) = \Lambda_k(L(\lambda)) - \alpha$.*

(c) *If $Q(\lambda) = \sum_{j=0}^{m} A_{m-j} \lambda^j$ then $\Lambda_k(Q(\lambda)) \setminus \{0\} = \left\{\mu^{-1} : \mu \in \Lambda_k(L(\lambda))\right\}$.*

(d) *If A_i, $i = 0, \ldots, m$ have a common totally isotropic subspace $\mathcal{S} = span\{x_1, \ldots, x_k\}$ with orthonormal vectors $x_j \in \mathbb{C}^n$, $j = 1, \ldots, k$, i.e. $x_l^* A_i x_j = 0$ for any $l, j = 1, \ldots, k$ and $i = 0, \ldots, m$, then $\Lambda_k(L(\lambda)) = \mathbb{C}$.*

Proposition 2. *Let $L(\lambda) = \sum_{j=0}^{m} A_j \lambda^j$ be a matrix polynomial, the following are equivalent:*

(i) *$\mu \in \Lambda_k(L(\lambda))$*

(ii) *there exists $M \in \mathcal{M}_{n,k}(\mathbb{C})$ with $rank M = k$ such that $M^* L(\mu) M = 0_k$*

(iii) *there exists an $L(\mu)$-orthogonal k-dimensional subspace $\mathcal{K}$ of $\mathbb{C}^n$*

(iv) *there exist $\{u_i\}_{i=1}^{k}$ orthonormal vectors such that $u_j^* L(\mu) u_i = 0$ for every $i, j = 1, \ldots, k$*

(v) *there exists a k-dimensional subspace $\mathcal{K}$ of $\mathbb{C}^n$ such that $v^* L(\mu) v = 0$ for every $v \in \mathcal{K}$*

(vi) *there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that*

$$U^* L(\mu) U = \begin{bmatrix} 0_k & L_1(\mu) \\ L_2(\mu) & L_3(\mu) \end{bmatrix},$$

where $L_1(\lambda), L_2(\lambda)$ and $L_3(\lambda)$ are suitable matrix polynomials.

Proof. The arguments (i)-(vi) are equivalent, since $\mu \in \Lambda_k(L(\lambda))$ is equivalent to $0 \in \Lambda_k(L(\mu))$. Further, we refer to the Proposition 1.1 in [4]. $\square$

Proposition 3. *Let* $L(\lambda) = \sum_{j=1}^{m} A_j \lambda^j$, *then*

$$\Lambda_k(L(\lambda)) \subseteq \Lambda_{k-1}(L(\lambda)) \subseteq \ldots \subseteq \Lambda_1(L(\lambda)).$$

Proof. For any $j \in \{2, \ldots, k\}$, let $\mu_0 \in \Lambda_j(L(\lambda))$. Then $0 \in \Lambda_j(L(\mu_0)) \subseteq \Lambda_{j-1}(L(\mu_0))$ and consequently, by $0 \in \Lambda_{j-1}(L(\mu_0))$, we conclude that $\mu_0 \in \Lambda_{j-1}(L(\lambda))$. $\square$

Corollary 4. *Let* $L(\lambda)$ *be an* $n \times n$ *matrix polynomial. Then for any* $k \leq n$

$$\Lambda_k(\underbrace{L(\lambda) \oplus \ldots \oplus L(\lambda)}_{k}) = w(L(\lambda)),$$

i.e. $\Lambda_k(\oplus_k L(\lambda))$ *is a non-empty set.*

Proof. Due to Proposition 3, $\mu_0 \in \Lambda_k(\oplus_k L(\lambda)) \subseteq w(\oplus_k L(\lambda))$. Hence $0 \in F(\oplus_k L(\mu_0)) = F(L(\mu_0))$, equivalently $\mu_0 \in w(L(\lambda))$ and then we obtain $\Lambda_k(\oplus_k L(\lambda)) \subseteq w(L(\lambda))$. In addition, $\mu_0 \in w(L(\lambda)) \Rightarrow 0 \in F(L(\mu_0)) = \cap_k F(L(\mu_0)) \subseteq \Lambda_k(\oplus_k L(\mu_0))$, according to a relation in [7]. Thus $w(L(\lambda)) \subseteq \Lambda_k(\oplus_k L(\lambda))$ and the proof is established. $\square$

The following result sketches the higher rank numerical range of a *square matrix* through numerical ranges.

Theorem 5. *Let* $A \in \mathcal{M}_n(\mathbb{C})$. *Then*

$$\Lambda_k(A) = \bigcap_M F(M^* A M),$$

where M *is any* $n \times (n-k+1)$ *isometry.*

The preceding expression of $\Lambda_k(A)$ indicates the "*convexity of* $\Lambda_k(A)$" in another way, since the Toeplitz-Hausdorff theorem ensures that each $F(M^*AM)$ is convex. For $k = n$, clearly $\Lambda_n(A) = \bigcap_{x \in \mathbb{C}^n, \|x\|=1} F(x^*Ax)$ and should be $\Lambda_n(A) \neq \emptyset$ *precisely* when A is scalar.

By Theorem 5, we may also describe $\Lambda_k(A)$ as intersections of circular discs as in [2, 3], i.e.

$$\Lambda_k(A) = \bigcap_M \left\{ \bigcap_{\gamma \in \mathbb{C}} \mathcal{D}\left(\gamma, \|M^*AM - \gamma I_{n-k+1}\|_2\right) \right\}.$$

Since $\Lambda_k(I\lambda - A)$ is identified with the higher rank numerical range of a matrix $A \in \mathcal{M}_n(\mathbb{C})$, Theorem 5 paves also the way for a characterization of $\Lambda_k(L(\lambda))$, demonstrated in the next proposition.

Proposition 6. *Suppose* $L(\lambda) = \sum_{j=1}^{m} A_j \lambda^j$, *then*

$$\Lambda_k(L(\lambda)) = \bigcap_M w(M^* L(\lambda) M) = \bigcup_N \Lambda_k(N^* L(\lambda) N),$$

where $M \in \mathcal{M}_{n,n-k+1}(\mathbb{C})$, $N \in \mathcal{M}_{n,k}(\mathbb{C})$ *are isometries.*

Proof. Obviously, by Theorem 5

$$\mu_0 \in \Lambda_k(L(\lambda)) \Leftrightarrow 0 \in \Lambda_k(L(\mu_0)) \Leftrightarrow$$

$$0 \in \bigcap_M F(M^* L(\mu_0) M) \Leftrightarrow \mu_0 \in \bigcap_M w(M^* L(\lambda) M).$$

Evenly, considering the equation $\Lambda_k(A) = \bigcup_N \Lambda_k(N^* A N)$ [1], we have

$$\mu_0 \in \Lambda_k(L(\lambda)) \Leftrightarrow 0 \in \Lambda_k(L(\mu_0)) \Leftrightarrow$$

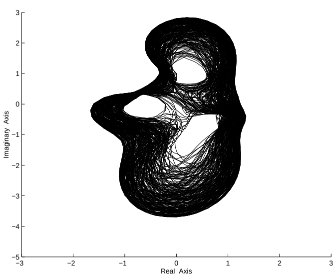
$$0 \in \bigcup_N \Lambda_k(N^* L(\mu_0) N) \Leftrightarrow \mu_0 \in \bigcup_N \Lambda_k(N^* L(\lambda) N).$$

$\square$

We should note that Proposition 6 provides us an estimation of the boundary of $\Lambda_k(L(\lambda))$ through the numerical approximation of the numerical range $w(L(\lambda))$. Although the higher rank numerical range $\Lambda_k(I\lambda - A)$ is always connected and convex [20, 23], $\Lambda_k(L(\lambda))$ need *not satisfy* these properties, as we will see in the next example.

Example 1. Let

$$L(\lambda) = 3I_5\lambda^3 + \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ i & 2i & 3i & 4i & 5i \\ -2 & 1 & 2 & 1 & 2 \\ 0.3 & 0 & 0 & 0 & 0 \end{bmatrix} \lambda^2 + \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 \\ 0 & 4 & 5 & 6 & 0 \\ 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 7 & 8 \end{bmatrix} \lambda + \begin{bmatrix} 4 & -i & 1 & 0 & -2 \\ i & 2i & -6i & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ -i & 3i & 0 & 2 & 4 \\ 3 & 1 & 2 & 4 & 5 \end{bmatrix}$$



The intersection of the numerical ranges $w(M^* L(\lambda) M)$ by 400 randomly chosen 5×4 isometries M, approximates the set $\Lambda_2(L(\lambda))$ and it is illustrated by the areas of "white" holes inside the figure. Note that all figure constitutes the numerical range $w(L(\lambda))$.

Investigating the non emptyness of $\Lambda_k(L(\lambda))$, it is noticed that the necessary and sufficient condition $n \geq 3k - 2$ for $\Lambda_k(A) \neq \emptyset$ of $A \in \mathcal{M}_n$ [18] fails in general for matrix polynomials, as shown in the next two results. The first proposition refers to the emptyness of the set $\Lambda_k(A\lambda + B)$, where A, B are $n \times n$ complex hermitian matrices.

Proposition 7. *Let the $n \times n$ selfadjoint pencil $L(\lambda) = A\lambda + B$ such that $w(M^*(A\lambda + B)M) \neq \mathbb{C}$ for any $n \times (n-k+1)$ isometry M. If A is a positive semidefinite matrix where the algebraic multiplicity of the eigenvalue $\mu_A = 0$ is greater than $k-1$ and B is positive (or negative) definite, then $\Lambda_k(A\lambda + B) = \emptyset$, for any $k = 2, 3, \ldots, n$.*

Proof. Suppose B is a positive definite matrix. Due to the condition of the multiplicity of $\mu_A = 0$, the matrices M^*AM and M^*BM are positive semidefinite and positive definite, respectively, for any $n \times (n-k+1)$ isometry M. Moreover, $w(M^*(A\lambda+B)M) \neq \mathbb{C}$ and $w(M^*(A\lambda+B)M) = (-\infty, -\frac{1}{\nu_M}]$, [21, Th.9], where ν_M is the maximum eigenvalue of $(M^*BM)^{-1}M^*AM$. Then, by Proposition 6, we obtain

$$\Lambda_k(A\lambda + B) = \bigcap_M w(M^*(A\lambda + B)M) = \bigcap_M (-\infty, -\frac{1}{\nu_M}] = \mathbb{R}^c = \emptyset.$$

Similarly, if B is a negative definite matrix. □

Moreover, in the next proposition $\Lambda_k(L(\lambda))$ appears to be non empty, with $L(\lambda)$ of special form.

Proposition 8. *Let $L(\lambda) = (\lambda - \lambda_0)^m A_m$ be an $n \times n$ matrix polynomial, where $A_m \neq 0$ and $0 \notin \Lambda_k(A_m)$. Then $\Lambda_k(L(\lambda))$ is a singleton, i.e. $\Lambda_k(L(\lambda)) = \{\lambda_0\}$, $\lambda_0 \in \mathbb{C}$.*

Proof. Since $0 \notin \Lambda_k(A_m)$, by Theorem 5, there exists an $n \times (n-k+1)$ isometry M_0 such that $0 \notin F(M_0^* A_m M_0)$ and evenly $w(M_0^* L(\lambda) M_0) = w((\lambda - \lambda_0)^m M_0^* A_m M_0) = \{\lambda_0\}$. Due to the special form of $M^*L(\lambda)M$, $\lambda_0 \in w(M^*L(\lambda)M)$ for all $n \times (n-k+1)$ isometries M, whereupon by Proposition 6, we have

$$\Lambda_k(L(\lambda)) = \bigcap_M w(M^*L(\lambda)M) = w(M_0^*L(\lambda)M_0) = \{\lambda_0\}.$$

□

In order to obtain $\Lambda_k(L(\lambda)) \neq \emptyset$ for any matrix polynomial $L(\lambda) = \sum_{l=0}^m A_l \lambda^l$ with $A_m \neq 0$, we are led to the common roots of the $k^2 > 1$ scalar polynomials $b_{ij}(\lambda, Q) = q_i^* L(\lambda) q_j$, $i, j = 1, \ldots, k$ for some isometries $Q = \begin{bmatrix} q_1 & \ldots & q_k \end{bmatrix} \in \mathcal{M}_{n,k}$. Adapting the notion of the Sylvester matrix R_s appeared in [15] and the discussion therein to the polynomials

$$\begin{aligned} b_{ij}(\lambda, Q) &= q_i^* A_m q_j \lambda^m + \ldots + q_i^* A_l q_j \lambda^l + \ldots + q_i^* A_0 q_j \\ &= b_{ij}^{(m)}(Q)\lambda^m + \ldots + b_{ij}^{(l)}(Q)\lambda^l + \ldots + b_{ij}^{(0)}(Q) \end{aligned} \tag{2.1}$$

for all $i, j = 1, \ldots, k$ and for some $n \times k$ isometry $Q = \begin{bmatrix} q_1 & \ldots & q_k \end{bmatrix}$, we have a condition for the polynomials $b_{ij}(\lambda, Q)$ to share polynomial common

factors. Denote by $\sigma \leq m$ to be the largest degree of the k^2 polynomials $b_{ij}(\lambda, Q)$ and let, as in (2.1)

$$b_{i_1,j_1}(\lambda, Q) = b^{(\sigma)}_{i_1,j_1}(Q)\lambda^\sigma + \ldots + b^{(l)}_{i_1,j_1}(Q)\lambda^l + \ldots + b^{(0)}_{i_1,j_1}(Q), \tag{2.2}$$

for some indices $i_1, j_1 \in \{1, \ldots, k\}$. If $\tau \leq \sigma$ is the largest degree of the remaining polynomials, then the generalized Sylvester matrix is

$$R_s(Q) = \begin{bmatrix} R_1(Q) \\ \vdots \\ R_{k^2}(Q) \end{bmatrix}, \tag{2.3}$$

where $R_1(Q)$ is the stripped $\tau \times (\sigma + \tau)$ matrix

$$R_1(Q) = \begin{bmatrix} b^{(\sigma)}_{i_1,j_1}(Q) & b^{(\sigma-1)}_{i_1,j_1}(Q) & & \cdots & b^{(0)}_{i_1,j_1}(Q) & & \mathbf{0} \\ & b^{(\sigma)}_{i_1,j_1}(Q) & b^{(\sigma-1)}_{i_1,j_1}(Q) & & & & \\ & \ddots & & \ddots & & \ddots & \\ \mathbf{0} & & b^{(\sigma)}_{i_1,j_1}(Q) & \cdots & b^{(\sigma-1)}_{i_1,j_1}(Q) & \cdots & b^{(0)}_{i_1,j_1}(Q) \end{bmatrix}$$

and for $p = 2, \ldots, k^2$, $R_p(Q)$ are the following $\sigma \times (\sigma + \tau)$ matrices

$$R_p(Q) = \begin{bmatrix} \mathbf{0} & & b^{(\tau)}_{i_p,j_p}(Q) & \cdot & \cdot & b^{(0)}_{i_p,j_p}(Q) \\ & b^{(\tau)}_{i_p,j_p}(Q) & & & & \\ & \cdot & \cdot & \cdot & \cdot & \\ b^{(\tau)}_{i_p,j_p}(Q) & \cdot & \cdot & b^{(0)}_{i_p,j_p}(Q) & & \mathbf{0} \end{bmatrix}$$

with $i_p, j_p \in \{1, \ldots, k\}$ and $i_p \neq i_1$, $j_p \neq j_1$. Hence, the degree $\delta(Q) \neq 0$ of the greatest common divisor of $b_{ij}(\lambda, Q)$ $(i, j = 1, \ldots, k)$ for some $n \times k$ isometry Q satisfies the relation

$$rank R_s(Q) = \tau + \sigma - \delta(Q) \leq 2m - \delta(Q) \tag{2.4}$$

and clearly, $\Lambda_k(L(\lambda)) \neq \emptyset$ if and only if there exists an $n \times k$ isometry Q such that $\text{rank} R_s(Q) < 2m$.

Following, we investigate the boundedness of $\Lambda_k(L(\lambda))$ and we state the next helpful lemma.

Lemma 9. *Let $A \in \mathcal{M}_n(\mathbb{C})$ with $A \neq 0$. For the function $f : \mathcal{M}_{n,k}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ defined by $f(Q) = Q^*AQ$, we have int(ker$f$) $= \emptyset$.*

Proof. For $k = 1$, let $\text{int}(\ker f) \neq \emptyset$ and a vector $x_0 \in \mathbb{C}^n \cap \text{int}(\ker f)$. Then there exists an open ball $\mathcal{B}(x_0, \varepsilon) \subset \ker f$ with $\varepsilon > 0$. For any $y \in \mathbb{C}^n$ with $y \in \mathcal{B}(0, \varepsilon)$ and real $t < 1$, clearly $ty \in \mathcal{B}(0, t\varepsilon) \subset \mathcal{B}(0, \varepsilon)$ and $x_0 + ty \in \mathcal{B}(x_0, \varepsilon)$. Hence,

$$f(x_0 + y) = f(x_0 + ty) = 0$$

and consequently we have $(t^2 - t)y^*Ay = 0$ for any $y \in \mathcal{B}(0, \varepsilon)$. Therefore, $A = 0$, which is a contradiction.

For $k > 1$, suppose $Q_0 \in \mathcal{M}_{n,k}(\mathbb{C}) \cap \mathrm{int}(\ker f)$ and let the open ball $\mathcal{B}(Q_0, \varepsilon) \subset \ker f$. If an $n \times k$ matrix $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_k \end{bmatrix} \in \mathcal{B}(Q_0, \varepsilon)$ and denote $Q_0 = \begin{bmatrix} q_{01} & q_{02} & \dots & q_{0k} \end{bmatrix}$, then

$$\|q_i - q_{0i}\|_2 = \|(Q - Q_0)e_i\|_2 \leq \|Q - Q_0\|_2 < \varepsilon \tag{2.5}$$

for $i = 1, \dots, k$, where $e_i \in \mathbb{C}^n$ is the i-th vector of the standard basis of $\mathbb{C}^n$ and $\|\cdot\|_2$ is the spectral norm. Hence, by $Q^*AQ = Q_0^*AQ_0 = 0$ we obtain $f(q_i) = q_i^*Aq_i = 0$ and $f(q_{0i}) = q_{0i}^*Aq_{0i} = 0$ $(i = 1, \dots, k)$ and by (2.5) we conclude $q_i \in \mathcal{B}(q_{0i}, \varepsilon)$, i.e. $\mathcal{B}(q_{0i}, \varepsilon) \subset \ker f$. This contradicts the emptyness of $\mathrm{int}(\ker f)$ in the vector case. □

Proposition 10. *Let $L(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, where $A_m \neq 0$. If $0 \notin \Lambda_k(A_m)$, then $\Lambda_k(L(\lambda)) \neq \emptyset$ is bounded.*

Conversely, assume that rank$R_s(Q) < 2m$, where $R_s(Q)$ is the Sylvester matrix in (2.3) of k^2 scalar polynomials, elements of matrix $Q^*L(\lambda)Q$, for all isometries $Q \in \mathcal{M}_{n,k}$ such that $Q^*A_mQ = zI_k$ $(z \in \mathbb{C}\setminus\{0\})$. If $\Lambda_k(A_m) \neq \{0\}$ and $\Lambda_k(L(\lambda))$ is bounded, then $0 \notin \Lambda_k(A_m)$.

Proof. Initially, we should remark that we investigate the boundedness of $\Lambda_k(L(\lambda))$ taking into account the condition (2.4), so that it is not empty and all the sets $\Lambda_1(L(\lambda)) \supseteq \ldots \supseteq \Lambda_{k-1}(L(\lambda))$ are not bounded. If $0 \notin \Lambda_k(A_m)$, then by Theorem 5 there exists an $n \times (n-k+1)$ isometry M_0 such that $0 \notin F(M_0^*A_mM_0)$. Hence, $w(M_0^*L(\lambda)M_0)$ is bounded [14] and by Proposition 6, as $\Lambda_k(L(\lambda)) \subseteq w(M_0^*L(\lambda)M_0)$, we conclude that $\Lambda_k(L(\lambda))$ is bounded.

For the converse, suppose that $0 \in \Lambda_k(A_m) \neq \{0\}$ and $\Lambda_k(L(\lambda))$ is bounded. We may find a sequence $\{z_\nu\} \subseteq \Lambda_k(A_m)$ such that $\lim_{\nu\to\infty} z_\nu = 0$ and consequently, a sequence of $n \times k$ isometries $\{Q_\nu\}$ such that $Q_\nu^*A_mQ_\nu = z_\nu I_k \to 0_k$. Due to the compactness of the group of $n \times k$ isometries, there is a subsequence $\{Q_\rho\}$ of $\{Q_\nu\}$ such that $\lim_{\rho\to\infty} Q_\rho = Q_0$, with $Q_0 \in \mathcal{M}_{n,k}$ be an isometry. Hence, by continuity, $\lim_{\rho\to\infty} Q_\rho^*A_mQ_\rho = Q_0^*A_mQ_0 = 0_k$ and by Lemma 9, should be $Q_\rho^*A_mQ_\rho = z_\rho I_k \neq 0$. Note that in (2.3), the Sylvester matrix $R_s(Q_\rho)$ has dimensions $k^2m \times 2m$, since in (2.2), $\sigma = \tau = m$ and due to rank$R_s(Q_\rho) < 2m$, the equation $Q_\rho^*L(\lambda)Q_\rho = 0_k$ always guarantees roots.

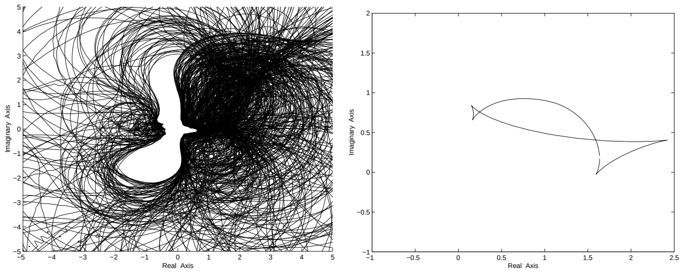
Moreover, there exists an index $j \neq m$ such that $Q_0^*A_jQ_0 \neq 0_k$ (otherwise $\Lambda_k(L(\lambda)) \equiv \mathbb{C}$) and evenly, $\|Q_\rho^*A_jQ_\rho\| \geq \varepsilon$ for some fixed $\varepsilon > 0$ and sufficiently large ρ. Hence, the $(m-j)$th elementary symmetric function $\pm\frac{1}{z_\rho}Q_\rho^*A_jQ_\rho$ of the roots of the matrix polynomial $Q_\rho^*L(\lambda)Q_\rho$ [8, Th.4.2] is not bounded, concluding that $\Lambda_k(L(\lambda))$ is not bounded. This contradicts the assumption and the proof is complete. □

Obviously, if $L(\lambda)$ is a monic matrix polynomial, then $\Lambda_k(L(\lambda))$ is always bounded. Following, we present an illustrative example of Proposition 10.

Example 2.

I. Let the matrix polynomial

$$L(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 2 & i & 0 & 2 \\ -i & 0 & -2 & 8 \end{bmatrix} \lambda^2 + \begin{bmatrix} i & 2 & i & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ i & 0 & i & 0 \end{bmatrix} \lambda + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$



The uncovered area in the left picture approximates the set $\Lambda_2(L(\lambda))$, which is bounded, although $\Lambda_1(L(\lambda)) = \mathbb{C}$. The boundary of $\Lambda_2(A_2)$ of the leading coefficient A_2 is illustrated on the right and we observe that $0 \notin \Lambda_2(A_2)$.

II. For the converse, let the 4×4 matrix polynomial ($m = 1$)

$$L(\lambda) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_1\lambda + A_0.$$

Firstly, we observe that $0 \in \Lambda_2(A_1) = [0, 3]$. On the other hand, $\Lambda_2(L(\lambda))$ is equal to the bounded set $\{0\}$. In fact, if we take the 4×3 isometries $M_1 = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$, then $M_1^* A_1 M_1$ and $M_1^* A_0 M_1$ are both positive semidefinite matrices and consequently, [21, Th.9], $w(M_1^* L(\lambda) M_1) = (-\infty, 0]$. Similarly, $M_2^* A_1 M_2$, $M_2^* A_0 M_2$ are positive and negative semidefinite, respectively, which verifies $w(M_2^* L(\lambda) M_2) = [0, \infty)$. Clearly, $\Lambda_2(L(\lambda)) \subseteq w(M_1^* L(\lambda) M_1) \cap w(M_2^* L(\lambda) M_2) = \{0\}$ and $0 \in \Lambda_2(L(\lambda)) \neq \emptyset$, i.e. $\Lambda_2(L(\lambda)) = \{0\}$.

In addition, for the isometry $Q = \begin{bmatrix} 0 & 1/\sqrt{3} \\ -\sqrt{6}/4 & 1/\sqrt{3} \\ \sqrt{6}/4 & 1/\sqrt{3} \\ 1/2 & 0 \end{bmatrix}$ we have $Q^* A_1 Q = I_2$ and in (2.3) the Sylvester matrix $R_s(Q) = \begin{bmatrix} 1 & 3/8 \\ 1 & 1/3 \\ 0 & -3\sqrt{2}/4 \\ 0 & -3\sqrt{2}/4 \end{bmatrix}$ has $\operatorname{rank} R_s(Q) = 2$, not less than 2, as it is required.

III. Consider the 4×4 matrix polynomial $L(\lambda) = I_2 \otimes (B\lambda + I_2)$, with

$B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $\Lambda_2(I_2 \otimes B) \neq \{0\}$ and additionally, $0 \in \Lambda_2(I_2 \otimes B)$. In this case, for any 4×2 isometry Q such that $Q^*(I_2 \otimes B)Q = zI_2 \neq 0_2$, the Sylvester matrix in (2.3) is $R_s(Q) = \left[\begin{smallmatrix} 1 & 1/z \\ 1 & 1/z \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\mathrm{rank} R_s(Q) = 1 < 2$. Since, $0 \in F(A_2)$, then $w(L(\lambda))$ as well as $\Lambda_2(L(\lambda) \oplus L(\lambda))$ (Corollary 4) are unbounded. It was expected by the converse of Proposition 10.

Further, we study the connectedness of $\Lambda_k(L(\lambda))$, attempting to specify a bound for the number of its connected components.

Proposition 11. *Let $L(\lambda) = A_m\lambda^m + \ldots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, with $A_m \neq 0$ and let $\Lambda_k(L(\lambda)) \neq \emptyset$ have ρ connected components. Moreover, $\mathrm{rank} R_s(Q) < 2m$, where $R_s(Q)$ is the Sylvester matrix in* (2.3) *of k^2 polynomials (elements of $Q^*L(\lambda)Q$), for any $n \times k$ isometry Q such that $Q^*A_mQ = \gamma I_k$ with $\gamma \in \Lambda_k(A_m) \setminus \{0\}$.*

If $\Lambda_k(A_m) \setminus \{0\}$ is connected, then $\rho \leq l \leq m$, where l is the minimum number of distinct roots of the equation $Q^*L(\lambda)Q = 0$ for any $n \times k$ isometry Q such that $Q^*A_mQ = \gamma I_k$, with $\gamma \in \Lambda_k(A_m) \setminus \{0\}$.

Otherwise, if $\Lambda_k(A_m) \setminus \{0\} = \mathcal{C}_1 \cup \mathcal{C}_2$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ and $\mathcal{C}_i$, $i = 1, 2$ are connected, then $\rho \leq l_1 + l_2 \leq 2m$, where l_i is the minimum number of distinct roots of $Q^*L(\lambda)Q = 0$ for any $n \times k$ isometry Q that corresponds to points $\gamma \in \mathcal{C}_i$, for $i = 1, 2$.

Proof. Let $\mathcal{C}_1$ be a connected component of $\Lambda_k(A_m) \setminus \{0\}$ and the $n \times k$ isometries $Q_0 = \begin{bmatrix} q_{01} & \ldots & q_{0k} \end{bmatrix}$, $Q_1 = \begin{bmatrix} q_{11} & \ldots & q_{1k} \end{bmatrix}$ correspond to $Q_0^*A_mQ_0 = \gamma_0 I_k$ and $Q_1^*A_mQ_1 = \gamma_1 I_k$, with $\gamma_0, \gamma_1 \in \mathcal{C}_1$. Evenly, we consider that $Q_0^*L(\lambda)Q_0 = 0_k$, $Q_1^*L(\lambda)Q_1 = 0_k$ and in particular, Q_0 has the property that provides the minimum number of distinct roots. We shall prove that there exists a continuous function of isometries $Q(t) : [0, 1] \to \mathcal{M}_{n,k}(\mathbb{C})$, with $Q(0) = Q_0$, $Q(1) = Q_1U$ for some unitary matrix U such that corresponds to a continuous path $\gamma(t) \in \mathcal{C}_1$ joining γ_0 to γ_1.

In case $\gamma_0 \neq \gamma_1$ and the line segment joining γ_0, γ_1 does not contain the origin, consider the continuous function

$$Q(t) = (\sqrt{1 - t^2}Q_0 + tQ_1U)C(t, U), \quad t \in [0, 1], \tag{2.6}$$

where $U = diag(e^{i\theta_1}, \ldots, e^{i\theta_k})$, with $\theta_j \in [0, 2\pi]$, $j = 1, \ldots, k$ and

$$C(t, U) = diag(c_1^{-1}(t, \theta_1), \ldots, c_k^{-1}(t, \theta_k)) \in \mathcal{M}_k,$$

where $c_j(t, \theta_j) = \|\sqrt{1 - t^2}q_{0j} + te^{i\theta_j}q_{1j}\|_2$, $j = 1, \ldots, k$. Clearly, $Q(0) = Q_0$, $Q(1) = Q_1U$ and $Q^*(t)Q(t) = I_k$, since the subspaces $\mathcal{K}_j = span\,\{q_{0j}, q_{1j}\}$ are pairwise orthogonal for all $j = 1, \ldots, k$ [11, p.318]. Hence, after some manipulations we obtain

$$Q^*(t)A_mQ(t) = C(t, U)\left[\gamma(t)I_k + t\sqrt{1 - t^2}(Q_0^*A_mQ_1U + U^*Q_1^*A_mQ_0)\right]C(t, U),$$

where $\gamma(t) = \gamma_0 + t^2(\gamma_1 - \gamma_0)$ for $t \in [0,1]$. Moreover, according to the conditions (i)-(iii) in the proof of Theorem 2.2 in [14], we may have a suitable unitary matrix $U_0 = diag(e^{i\theta_{01}}, \dots, e^{i\theta_{0k}})$ such that the matrix function

$$g(U) = Q_0^* A_m Q_1 U + U^* Q_1^* A_m Q_0$$

satisfies one of the following conditions:

(i) $g(U_0) = 0_k$,

(ii) $g(U_0) = \xi(\gamma_1 - \gamma_0)I_k$ for some real $\xi \neq 0$.

Then, $Q^*(t)A_m Q(t) = \left[\gamma_0 + (t^2 + \xi t\sqrt{1-t^2})(\gamma_1 - \gamma_0)\right] C^2(t, U_0) \neq 0_k$ and for all $j = 1, \dots, k$ the line segments $h_j(t) = \frac{\gamma_0 + (t^2 + \xi t\sqrt{1-t^2})(\gamma_1 - \gamma_0)}{c_j^2(t, \theta_{0j})} \neq 0$ join the points γ_0, γ_1 without these necessarily be endpoints. Apparently, due to the convexity of $\Lambda_k(A_m)$, we have that the isometries $Q(t)$ generate the line segment $\gamma(t) \in \mathcal{C}_1$.

In case the origin belongs to the line segment $[\gamma_0, \gamma_1]$, $(\gamma_0 \neq \gamma_1)$, we may consider another $\gamma_2 \in \mathcal{C}_1$ such that $\gamma_2 \neq \gamma_0, \gamma_1$ and $[\gamma_0, \gamma_2] \cup [\gamma_2, \gamma_1] \subseteq \mathcal{C}_1$. This is true because of the convexity of $\Lambda_k(A_m)$ and the fact that the points γ_0, γ_1 belong to the same connected component.

Finally, if $\gamma_0 = \gamma_1$ and A_m is a scalar matrix, then instead of (2.6) consider the continuous function of $n \times k$ isometries

$$Q(t) = (\sqrt{1-t^2} Q_0 + tQ_1)C(t, I_k), \quad t \in [0,1].$$

Otherwise, if A_m is not scalar, we refer to (2.6).

Thus, we have constructed a continuous function of $n \times k$ isometries $Q(t)$ such that $Q^*(t)A_m Q(t) = \gamma(t)I_k \neq 0_k$, $t \in [0,1]$ and this asserts that the Sylvester matrix $R_s(Q(t)) \in \mathcal{M}_{k^2m, 2m}$ for $t \in [0,1]$, since $\sigma = \tau = m$ in (2.2). Hence, by the assumption $\text{rank} R_s(Q(t)) < 2m$ for all $t \in [0,1]$, we have that the equation $Q^*(t)L(\lambda(t))Q(t) = 0$ has roots, let $\lambda_1(t), \dots, \lambda_r(t)$ $(r \leq m)$. Due to the continuity of $Q(t)$, the roots $\lambda_j(t) : [0,1] \to \Lambda_k(L(\lambda))$ are continuous paths in $\Lambda_k(L(\lambda))$, connecting the roots of equations $Q_0^* L(\lambda) Q_0 = 0_k$ and $Q_1^* L(\lambda) Q_1 = 0_k$ and thus the proof is completed. □

Example 3.
Let the 4×4 quadratic matrix polynomial

$$L(\lambda) = \left[\begin{smallmatrix} 2i & 0 \\ 0 & -2i \end{smallmatrix}\right] \otimes I_2 \lambda^2 + 4I_4 \lambda = \lambda(D\lambda + 4I_4), \ \textit{with} \ D = \left[\begin{smallmatrix} 2i & 0 \\ 0 & -2i \end{smallmatrix}\right] \otimes I_2.$$

Obviously, $\Lambda_2(L(\lambda)) = \{0\} \cup \Lambda_2(D\lambda + 4I_4)$ and $0 \notin \Lambda_2(D\lambda + 4I_4) \neq \emptyset$, that is $\{0\}$ is an isolated point. We also note that $\mu_0 \in \Lambda_2(D\lambda + 4I_4)$ if and only if $\mu_0^{-1} \in \Lambda_2(\left[\begin{smallmatrix} -i/2 & 0 \\ 0 & i/2 \end{smallmatrix}\right] \otimes I_2) \setminus \{0\}$, therefore $\Lambda_2(L(\lambda))$ has three

connected components, two on the imaginary axis, the sets $(-\infty,-2]$, $[2,\infty)$ and $\{0\}$. Moreover, for the 8×4 Sylvester matrix $R_s(Q)$ in (2.3) we have $\operatorname{rank} R_s(Q) = \operatorname{rank}\begin{bmatrix} \lambda_0 & 0 & 4 & 0 \\ 0 & \lambda_0 & 0 & 4 \\ \lambda_0 & 0 & 4 & 0 \\ 0 & \lambda_0 & 0 & 4 \end{bmatrix} < 4$ for all isometries $Q \in \mathcal{M}_{4,2}$ such that $Q^*DQ = \lambda_0 I_2 \neq 0_2$. Also $\Lambda_2(D)\setminus\{0\}$ has two connected components and Proposition 11 is confirmed.

3 Sharp points

In this section, following [16], we define the notion of sharp points. Particularly, $z_0 \in \partial\Lambda_k(L(\lambda))$ is called to be a *sharp point* if for a connected component $\Lambda_k^{(s)}(L(\lambda))$ of $\Lambda_k(L(\lambda))$ there exist a disc $S(z_0,\varepsilon)$, with $\varepsilon > 0$ and two angles $\theta_1 < \theta_2$, with $\theta_1,\theta_2 \in [0,2\pi)$, such that

$$\operatorname{Re}(e^{i\theta}z_0) = \max\left\{\operatorname{Re} z : e^{-i\theta}z \in \Lambda_k^{(s)}(L(\lambda)) \cap S(z_0,\varepsilon)\right\} \quad \forall\, \theta \in (\theta_1,\theta_2).$$

The following proposition presents a condition for a boundary point of $w(L(\lambda))$ to be a boundary point of $\Lambda_k(L(\lambda))$, as well. We should remark that the term 'multiplicity' as mentioned below is referred to the *algebraic multiplicity* of an eigenvalue.

Proposition 12. *Let the $n\times n$ matrix polynomial $L(\lambda)$. If $\gamma \in \sigma(L(\lambda)) \cap \partial w(L(\lambda))$ with multiplicity k, then for $j = 2,\dots,k$*

$$\gamma \in \partial\Lambda_j(L(\lambda)).$$

Proof. Clearly, by the assumption, γ is seminormal eigenvalue of the matrix polynomial $L(\lambda)$ of multiplicity k [13, Th.6]. That is, there exists a unitary matrix U such that

$$U^*L(\gamma)U = 0_k \oplus R(\gamma),$$

where $R(\lambda)$ is an $(n-k)\times(n-k)$ matrix polynomial and $\gamma \notin \operatorname{int} w(R(\lambda))$. Hence, by Propositions 2(vi) and 3, it is implied that $\gamma \in \Lambda_j(L(\lambda)) \subseteq \Lambda_{j-1}(L(\lambda))$ for $j = 2,\dots,k$ and due to $\gamma \notin \operatorname{int} w(L(\lambda))$ ($\equiv \operatorname{int}\Lambda_1(L(\lambda))$), we obtain $\gamma \in \partial\Lambda_j(L(\lambda))$, for $j = 2,\dots,k$. $\square$

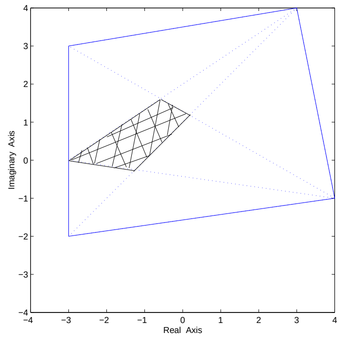
For the pencil $I\lambda - A$, we obtain the following corollary.

Corollary 13. *Let $A \in \mathcal{M}_n(\mathbb{C})$. If $\gamma \in \partial F(A)$ is eigenvalue of A of multiplicity k, then*

$$\gamma \in \partial\Lambda_j(A), \quad j = 2,\dots,k.$$

The converse of Corollary 13 and consequently of Proposition 12 is not true, as it is illustrated in the next example.

Example 4.
Let $A = diag(3+4i, 4-i, -3-2i, -3, -3+3i)$. The outer polygon of the figure is $F(A)$, whereas the inner shaded polygon is $\Lambda_2(A)$, which is the intersection of all $\binom{5}{4}$ convex combinations of the eigenvalues $\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}, \lambda_{j_4}$ of A, with $1 \leq j_1 \leq \ldots \leq j_4 \leq 5$. Notice that $\lambda_0 = -3 \in \partial F(A) \cap \partial\Lambda_2(A)$, but it is a simple eigenvalue of matrix A. In addition, $\Lambda_3(A) = \emptyset$.



In view of the definition of sharp points, for a pencil $A\lambda - B$, we have the next proposition.

Proposition 14. *Let the pencil $L(\lambda) = A\lambda - B \in \mathcal{M}_n(\mathbb{C})$ and z_0 be a sharp point of $w(A\lambda - B)$ of multiplicity k with respect to the spectrum $\sigma(A\lambda - B)$, then z_0 is also a sharp point of $\Lambda_j(A\lambda - B)$, for $j = 2, \ldots, k$.*

Proof. Since the sharp point z_0 of $w(A\lambda - B)$ is also an eigenvalue of the pencil $A\lambda - B$ [17, Th.1.3], with multiplicity k by hypothesis, we deduce by Proposition 12 that $z_0 \in \partial\Lambda_j(A\lambda - B)$, for $j = 2, \ldots, k$. It only suffices to prove that for any disc $S(z_0, \varepsilon)$ with $\varepsilon > 0$, z_0 satisfies the equality

$$\mathrm{Re}(e^{i\theta} z_0) = \max\left\{\mathrm{Re}\, z : e^{-i\theta} z \in \Lambda_j(A\lambda - B) \cap S(z_0, \varepsilon)\right\}$$

or equivalently, due to Proposition 6

$$\mathrm{Re}(e^{i\theta} z_0) = \max\left\{\mathrm{Re}\, z : z \in \bigcap_M \left(w(e^{i\theta} M^*(A\lambda - B)M) \cap S(e^{i\theta} z_0, \varepsilon)\right)\right\}$$

for every angle $\theta \in (\theta_1, \theta_2)$ with $0 \leq \theta_1 < \theta_2 < 2\pi$.

The inclusion relation $w(M^*(A\lambda - B)M) \subseteq w(A\lambda - B)$ for any $n \times (n - j + 1)$ isometry M, $j = 2, \ldots, k$ verifies the inequality

(3.1)
$$\max_{\bigcap_M \left(w(e^{i\theta} M^*(A\lambda - B)M) \cap S(e^{i\theta} z_0, \varepsilon)\right)} \mathrm{Re}\, z \leq \max_{w(e^{i\theta}(A\lambda - B)) \cap S(e^{i\theta} z_0, \varepsilon)} \mathrm{Re}\, z = \mathrm{Re}(e^{i\theta} z_0)$$

for any disc $S(e^{i\theta} z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$.

Moreover, $\ker(Az_0 - B) \cap \operatorname{Im}(MM^*) \neq \emptyset$, since $\dim\ker(Az_0 - B) + \dim\operatorname{Im}(MM^*) = k+n-j+1 \geq n+1$. Therefore, for an eigenvector $x_0 \in \mathbb{C}^n$ of $A\lambda - B$ corresponding to z_0 there exists a vector $y_0 \in \mathbb{C}^n$ such that $x_0 = MM^*y_0$. Obviously, $M^*y_0 \in \mathbb{C}^{n-j+1}$ is an eigenvector of $M^*(A\lambda - B)M$ corresponding to z_0, yielding $z_0 \in \sigma(M^*(A\lambda - B)M) \subseteq w(M^*(A\lambda - B)M)$ for any $n \times (n-j+1)$ isometry M.

Thus, $z_0 \in \bigcap_M w(M^*(A\lambda - B)M)$, i.e. $\operatorname{Re} z_0 \in \operatorname{Re}(\bigcap_M w(M^*(A\lambda - B)M))$, whereupon we confirm the inequality

$$\operatorname{Re}(e^{i\theta}z_0) \leq \max_{\bigcap_M \left(w(e^{i\theta}M^*(A\lambda-B)M)\cap S(e^{i\theta}z_0,\varepsilon)\right)} \operatorname{Re} z \tag{3.2}$$

for any disc $S(e^{i\theta}z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$. Therefore, by (3.1) and (3.2)

$$\operatorname{Re}(e^{i\theta}z_0) = \max\left\{\operatorname{Re} z : z \in \bigcap_M \left(w(e^{i\theta}M^*(A\lambda - B)M) \cap S(e^{i\theta}z_0, \varepsilon)\right)\right\}$$

for any disc $S(e^{i\theta}z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$, establishing the assertion. □

Because of the previous results, we obtain an interesting corollary concerning the sharp points of the higher rank numerical range of a matrix $A \in \mathcal{M}_n(\mathbb{C})$.

Corollary 15. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and $z_0 \in \partial F(A)$ be a sharp point of $F(A)$ of multiplicity k with respect to $\sigma(A)$, then z_0 is also a sharp point of $\Lambda_j(A)$, for $j = 2, \ldots, k$.*

Analogous statement to Proposition 14 for the "sharp points" of $\Lambda_j(L(\lambda))$ we may confirm taking into consideration Theorem 1.4 in [17].

4 Connection between $\Lambda_k(L(\lambda))$ and $\Lambda_k(\mathbf{A})$

Let the matrix polynomial $L(\lambda) = \sum_{i=0}^m A_i\lambda^i$ and the corresponding $mn \times mn$ companion pencil

$$C_L(\lambda) = \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & 0 \\ 0 & & \cdots & & A_m \end{bmatrix} \lambda - \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & I_n \\ A_0 & & \cdots & & A_{m-1} \end{bmatrix},$$

well known as *linearization* of $L(\lambda)$, since there exist suitable matrix polynomials $E(\lambda)$, $F(\lambda)$ with constant nonzero determinants such that

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{n(m-1)} \end{bmatrix} = E(\lambda)C_L(\lambda)F(\lambda).$$

Next, we generalize a corresponding relation in [16] between the higher rank numerical ranges of $L(\lambda)$ and $C_L(\lambda)$.

Proposition 16. $\Lambda_k(L(\lambda)) \cup \{0\} \subseteq \Lambda_k(C_L(\lambda))$.

Proof. By Proposition 6 and the relationship $w(L(\lambda)) \cup \{0\} \subseteq w(C_L(\lambda))$ in [16], we have

$$\Lambda_k(L(\lambda)) \cup \{0\} = \left(\bigcap_M w(M^* L(\lambda) M) \right) \cup \{0\} \subseteq \bigcap_M w(C_{M^*LM}(\lambda)), \tag{4.1}$$

where $M \in \mathcal{M}_{n,n-k+1}(\mathbb{C})$, with $M^*M = I_{n-k+1}$ and $C_{M^*LM}(\lambda)$ is the linearization of the matrix polynomial $M^*L(\lambda)M$. Since,

$$\begin{aligned} C_{M^*LM}(\lambda) &= (I_m \otimes M)^* \begin{bmatrix} \lambda I_n & -I_n & 0 & \cdots & 0 \\ 0 & \lambda I_n & -I_n & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & -I_n \\ A_0 & & \cdots & & A_m\lambda + A_{m-1} \end{bmatrix} (I_m \otimes M) \\ &= (I_m \otimes M)^* C_L(\lambda)(I_m \otimes M), \end{aligned}$$

considering the isometry $Q = \begin{bmatrix} I_m \otimes M & V \end{bmatrix} \in \mathcal{M}_{mn,mn-k+1}(\mathbb{C})$, with $Q^*Q = I_{mn-k+1}$, we have

$$\begin{aligned} \bigcap_M w(C_{M^*LM}(\lambda)) &= \bigcap_M w((I_m \otimes M)^* C_L(\lambda)(I_m \otimes M)) \\ &\subseteq \bigcap_Q w(Q^* C_L(\lambda) Q) \subseteq \bigcap_X w(X^* C_L(\lambda) X) = \Lambda_k(C_L(\lambda)), \end{aligned} \tag{4.2}$$

where $X \in \mathcal{M}_{mn,mn-k+1}(\mathbb{C})$ with $X^*X = I_{mn-k+1}$. Thus by (4.1) and (4.2) the proof is completed. $\square$

Furthermore, $\Lambda_k(L(\lambda))$ appears to be associated with the joint higher rank numerical range $\Lambda_k(\mathbf{A})$ of an $(m+1)$-tuple of $n \times n$ matrices $\mathbf{A} = (A_0, A_1, \dots, A_m)$. In fact,

$$\begin{aligned} \Lambda_k(L(\lambda)) &= \{\lambda \in \mathbb{C} : PA_mP\lambda^m + \ldots + PA_1P\lambda + PA_0P = 0_n\,,\ \ P \in \mathcal{P}_k\} \\ &\supseteq \{\lambda \in \mathbb{C} : (\mu_m\lambda^m + \ldots + \mu_1\lambda + \mu_0)P = 0_n\,,\ \ (\mu_0, \mu_1, \dots, \mu_m) \in \Lambda_k(\mathbf{A})\} \\ &= \{\lambda \in \mathbb{C} : \mu_m\lambda^m + \ldots + \mu_1\lambda + \mu_0 = 0\,,\ \ (\mu_0, \mu_1, \dots, \mu_m) \in \Lambda_k(\mathbf{A})\} \\ &= \{\lambda \in \mathbb{C} : \langle (1, \lambda, \dots, \lambda^m), \mathbf{u} \rangle = 0\,,\ \ \mathbf{u} = (\mu_0, \mu_1, \dots, \mu_m) \in \Lambda_k(\mathbf{A})\}\,. \end{aligned}$$

The above inclusion justifies that Q^*A_jQ may not be scalar matrices for $j = 0, \dots, m$ and for all isometries $Q \in \mathcal{M}_{n,k}(\mathbb{C})$.

The notion of the joint spectrum in [13], leads to an extension of Proposition 12.

Proposition 17. *Let* $\mathbf{A} = (A_0, \ldots, A_m)$ *be an* $(m+1)$*-tuple of* $n \times n$ *matrices. If* $(\mu_0, \ldots, \mu_m) \in \partial w(\mathbf{A})$ *is a normal joint eigenvalue of* $\mathbf{A}$ *with geometric multiplicity* k*, then*

$$(\mu_0, \ldots, \mu_m) \in \partial \Lambda_j(\mathbf{A}), \quad j = 2, \ldots, k.$$

Proof. Since $(\mu_0, \ldots, \mu_m)$ is a normal joint eigenvalue with geometric multiplicity k [13], there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that

$$(U^* A_0 U, \ldots, U^* A_m U) = (\mu_0 I_k \oplus B_0, \ldots, \mu_m I_k \oplus B_m),$$

where $(B_0, \ldots, B_m)$ is an $(m+1)$-tuple of $(n-k) \times (n-k)$ matrices and $(\mu_0, \ldots, \mu_m) \notin \sigma(B_0, \ldots, B_m)$. Thus, $(\mu_0, \ldots, \mu_m) \in \Lambda_k(\mathbf{A})$. Since the point $(\mu_0, \ldots, \mu_m) \in \partial w(\mathbf{A})$ and $\Lambda_j(\mathbf{A}) \subseteq \Lambda_{j-1}(\mathbf{A})$ for every $j = 2, \ldots, k$ [19], we establish $(\mu_0, \ldots, \mu_m) \in \partial \Lambda_j(\mathbf{A})$ for all $j = 2, \ldots, k$. □

Finally, we obtain the following result relative to that in [22].

Proposition 18. *Let the matrix polynomial* $L(\lambda) = \sum_{i=0}^m A_i \lambda^i$*. Then*

$$\Lambda_k(L(\lambda)) \supseteq \{\lambda \in \mathbb{C} : \langle (1, \lambda, \ldots, \lambda^m), \boldsymbol{u} \rangle = 0\,, \;\; \boldsymbol{u} \in co\Lambda_k(\mathbf{A})\}\,,$$

where $\mathbf{A} = (A_0, A_1, \ldots, A_m)$ *is the* $(m+1)$*-tuple of* $n \times n$ *matrices* A_i*.*

Proof. Let $\Omega = \{\lambda \in \mathbb{C} : \langle (1, \lambda, \ldots, \lambda^m), \mathbf{u} \rangle = 0\,, \mathbf{u} \in co\Lambda_k(\mathbf{A})\}$. To prove the inclusion, suppose $\lambda_0 \in \Omega$, that is

$$\langle (1, \lambda_0, \ldots, \lambda_0^m), \mathbf{u} \rangle = 0 \tag{4.3}$$

for some $\mathbf{u} = (u_0, u_1, \ldots, u_m) \in co\Lambda_k(\mathbf{A})$. We have $\Lambda_k(\mathbf{A}) \subseteq \mathbb{C}^{m+1} \equiv \mathbb{R}^{2m+2}$ and by Caratheodory's theorem in Convex Analysis, there are at most $2m+3$ elements of $\Lambda_k(\mathbf{A})$ such that

$$co\Lambda_k(\mathbf{A}) = \left\{ \sum_{j=1}^{\rho} \mu_j \mathbf{u}_j : \mathbf{u}_j \in \Lambda_k(\mathbf{A}), \; \mu_j \geq 0\,, \sum_{j=1}^{\rho} \mu_j = 1, \; with \; \rho \leq 2m+3 \right\}.$$

Hence, for $\mathbf{u} = (u_0, u_1, \ldots, u_m) \in co\Lambda_k(\mathbf{A})$ there are suitable $\mu_j \geq 0$, $\sum_j^{\rho} \mu_j = 1$, $\rho \leq 2m+3$ such that

$$\mathbf{u} = \mu_1 \mathbf{u}_1 + \ldots + \mu_\rho \mathbf{u}_\rho = \mu_1 \begin{bmatrix} u_{10} \\ \vdots \\ u_{1m} \end{bmatrix} + \ldots + \mu_\rho \begin{bmatrix} u_{\rho 0} \\ \vdots \\ u_{\rho m} \end{bmatrix}, \tag{4.4}$$

where $\mathbf{u}_j = \begin{bmatrix} u_{j0} & \ldots & u_{jm} \end{bmatrix}^T \in \Lambda_k(\mathbf{A})$, $j = 1, \ldots, \rho$ and by equations (4.3) and (4.4), we obtain:

$$\langle (1, \lambda_0, \ldots, \lambda_0^m), \mathbf{u} \rangle \;\; = \;\; \mu_1 \begin{bmatrix} 1 & \ldots & \lambda_0^m \end{bmatrix} \begin{bmatrix} u_{10} \\ \vdots \\ u_{1m} \end{bmatrix} + \ldots + \mu_\rho \begin{bmatrix} 1 & \ldots & \lambda_0^m \end{bmatrix} \begin{bmatrix} u_{\rho 0} \\ \vdots \\ u_{\rho m} \end{bmatrix}$$

i.e.

$$\mu_1 p_1(\lambda_0) + \ldots + \mu_\rho p_\rho(\lambda_0) = 0, \tag{4.5}$$

where $p_j(\lambda) = u_{jm}\lambda^m + \ldots + u_{j1}\lambda + u_{j0}$ for $j = 1, \ldots, \rho$. Evenly, by $\mathbf{u}_j = (u_{j0}, \ldots, u_{jm}) \in \Lambda_k(\mathbf{A})$, there exist rank-$k$ orthogonal projections P_j, $j = 1, \ldots, \rho$ such that $P_j A_i P_j = u_{ji} P_j$, $i = 0, \ldots, m$ and consequently

$$\begin{aligned} p_j(\lambda_0)P_j &= u_{j0}P_j + u_{j1}P_j\lambda_0 + \ldots + u_{jm}P_j\lambda_0^m \\ &= P_jA_0P_j + P_jA_1P_j\lambda_0 + \ldots + P_jA_mP_j\lambda_0^m \\ &= P_jL(\lambda_0)P_j, \end{aligned}$$

which means that $p_j(\lambda_0) \in \Lambda_k(L(\lambda_0))$. Due to the convexity of the higher rank numerical range of the matrix $L(\lambda_0)$ and the equation (4.5), $0 \in \Lambda_k(L(\lambda_0))$, equivalently $\lambda_0 \in \Lambda_k(L(\lambda))$. □

References

[1] Aik. Aretaki and J. Maroulas, Presentation at the 10th Workshop on "Numerical Ranges and Numerical Radii", Krakow, Poland, 2010.

[2] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Mathematical Society Lecture Note Series, Cambridge University Press, New York, 1971.

[3] F.F. Bonsall and J. Duncan, *Numerical Ranges II*, London Mathematical Society Lecture Notes Series, Cambridge University Press, New York, 1973.

[4] M.D. Choi, M. Giesinger, J.A. Holbrook and D.W. Kribs, Geometry of higher-rank numerical ranges, *Linear and Multilinear Algebra*, **56**(1), 53-64, 2008.

[5] M.D. Choi, J.A. Holbrook, D.W. Kribs and K. Zyczkowski, Higher-rank numerical ranges of unitary and normal matrices, preprint, http://arxiv.org/quant-ph/0608244.

[6] M.D. Choi, D.W. Kribs and K. Zyczkowski, Quantum error correcting codes from the compression formalism, *Reports on Mathematical Physics*, **58**, 77-86, 2006.

[7] M.D. Choi, D.W. Kribs and K. Zyczkowski, Higher-rank numerical ranges and compression problems, *Linear Algebra and its Applications*, **418**, 828-839, 2006.

[8] J.E. Dennis, Jr., J.F. Traub, R.P. Weber, The Algebraic Theory of Matrix Polynomials, *SIAM Journal on Numerical Analysis*, **13**, 831-845, 1976.

[9] I.Z. Emiris, E.D. Fritzilas and D. Manocha, Algebraic Algorithms for structure Determination in Biological chemistry, *International Journal of Quantum Chemistry*, **106**, 190-210, 2006.

[10] K.E. Gustafson and D.K.M. Rao, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer-Verlag, New York, 1997.

[11] P.R. Halmos, *A Hilbert Space Problem Book*, 2nd Ed., Springer-Verlag, New York, 1982.

[12] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[13] P. Lancaster and P. Psarrakos, Normal and seminormal eigenvalues of matrix functions, *Integral Equations and Operator Theory*, **41**, 331-342, 2001.

[14] C.K. Li, L. Rodman, Numerical range of matrix polynomials, *SIAM J. Matrix Analysis and Applications*, **15**, 1256-1265, 1994.

[15] J. Maroulas and D. Dascalopoulos, Applications of the Generalized Sylvester Matrix, *Applied Math. and Computations*, **8**, 121-135, 1981.

[16] J. Maroulas and P. Psarrakos, Geometrical Properties of numerical range of matrix polynomials, *Computers Math. Applic.*, **31**, 41-47, 1996.

[17] J. Maroulas and P. Psarrakos, The boundary of the numerical range of matrix polynomials, *Linear Algebra and its Applicatrions*, **267**, 101-111, 1997.

[18] C.K. Li, Y.T. Poon and N.S. Sze, Condition for the higher rank numerical range to be non-empty, *Linear and Multilinear Algebra*, **57**(4), 365-368, 2009.

[19] C.K Li and Y.T. Poon, Quantum error correction and generalized numerical ranges, preprint, http://arxiv.org/0812.4772v1 [math.FA], 2008.

[20] C.K. Li and N.S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations, *Proceedings of the American Mathematical Society*, **136**, 3013-3023, 2008.

[21] P.J. Psarrakos, Numerical range of linear pencils, *Linear Algebra and its Applicatrions*, **317**, 127-141, 2000.

[22] P.J. Psarrakos and M.L. Tsatsomeros, On the relation between the numerical range and the joint numerical range of matrix polynomials, *Electronic Journal of Linear Algebra*, **6**, 20-30, 2000.

[23] H.J. Woerdeman, The higher rank numerical range is convex, *Linear and Multilinear Algebra*, **56**(1), 65-67, 2007.